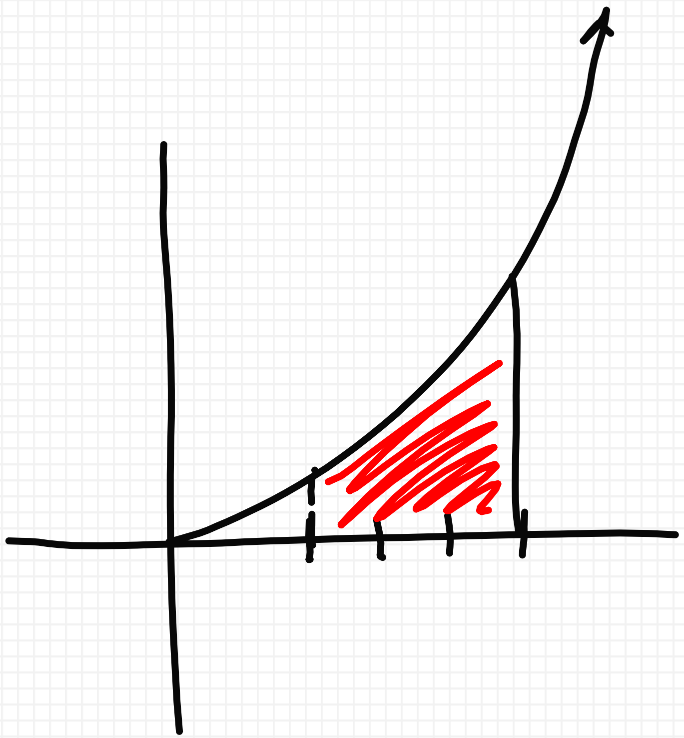
Fundamental Theorem(s) of Calculus

Section 5.3

Average Value of a Function:

IF f IS INTEGRABLE ON $[a,b]$

$$\Rightarrow \boxed{f_{AVE} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$



Ex: FIND f_{AVE} IF $f(x)=x^2$ ON $[1,4]$

$$f_{AVE} = \frac{1}{3}\int_1^4 x^2\,dx = \frac{1}{3}(21) = 7$$

Mean Value Theorem for Integrals:

IF f IS CONTINUOUS ON $[a,b]$ $\Longrightarrow$

$\exists$ $c \in [a,b]$ SUCH THAT

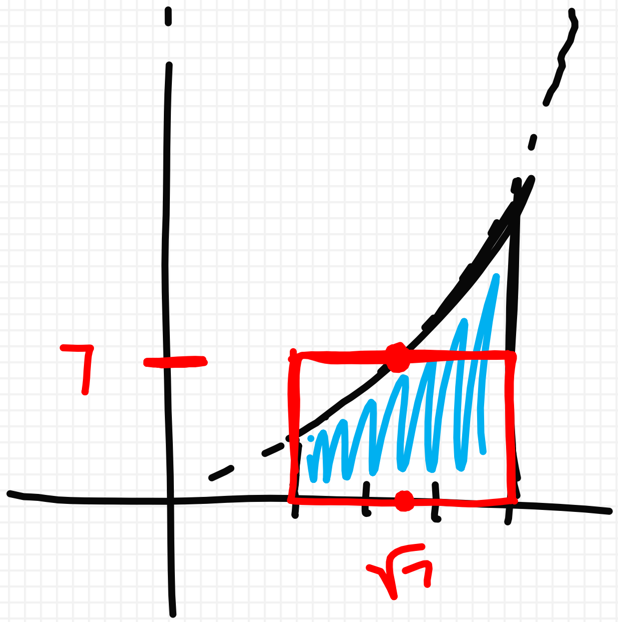
"THERE EXISTS"

$$f(c) = f_{AVE} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

FOR PREVIOUS EXAMPLE: $f(c) = 7$

$c^2 = 7$

$c = \sqrt{7}$



RED RECTAN. HAS SAME AREA AS BLUE REGION

Fundamental Theorem of Calculus, Part I: (FTC)

IF f IS CONTINUOUS ON $[a,b]$ THEN $F(x) = \int_a^x f(t)\,dt$

HAS A DERIVATIVE AT ALL $x \in [a,b]$ SUCH THAT

$$\frac{d}{dx}F(x) = \boxed{\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$$

→

Examples:

$$\frac{d}{dx}\underbrace{\int_1^x t^3\,dt} = \boxed{x^3}$$

$$\frac{d}{dx}\left(\frac{1}{4}t^4\Big]_1^x\right)$$

$$\frac{d}{dx}\left(\frac{1}{4}x^4 - \frac{1}{4}\right)$$

$$x^3$$

$$\frac{d}{dx}\int_1^x \frac{\sin t}{t}\,dt$$

$$= \frac{\sin x}{x}$$

Examples:

$$\frac{d}{dx}\int_3^{x^3} \sin t\,dt$$

$\sin(x^3)\cdot(3x^2)$

CHAIN RULE

$$\boxed{3x^2 \sin(x^3)}$$

$$\frac{d}{dx}\int_x^{x^2} \frac{1}{1+e^t}\,dt$$

$$\frac{d}{dx}\left(\int_0^{x^2} \frac{1}{1+e^t}\,dt - \int_0^{x} \frac{1}{1+e^t}\,dt\right)$$

$$\frac{d}{dx}\int_0^{x^2} \frac{1}{1+e^t}\,dt - \frac{d}{dx}\int_0^{x} \frac{1}{1+e^t}\,dt$$

$$\boxed{\frac{1}{1+e^{x^2}}\cdot 2x - \frac{1}{1+e^{x}}\cdot 1}$$

Fundamental Theorem of Calculus, Part I - Summary:

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

$$\frac{d}{dx}\int_x^a f(t)\,dt = \frac{d}{dx}\left[-\int_a^x f(t)\,dt\right] = -f(x)$$

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = \boxed{f(g(x))\cdot g'(x)}$$

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = \boxed{f(g(x))\cdot g'(x) - f(h(x))\,h'(x)}$$

AP Calculus BC
Section 5.4 – Fundamental Theorem of Calculus

Find $G'(x)$ if

1. $G(x)=\int_1^x 2t\,dt$

2. $G(x)=\int_x^1 3t^2\,dt$ $= -3x^2$

3. $G(x)=\int_0^x \left(2t^2+\sqrt{t}\right)dt$

4. $G(x)=\int_0^{x^3} \cos(2t)\,dt$ $= \cos(2x^3)\cdot 3x^2$

5. $G(x)=\int_3^{x^2+x} \sqrt{2t+\sin t}\,dt$

6. $G(x)=\int_{-x}^{x^2} \left(t^3-5\right)dt$ $\left((x^2)^3-5\right)(2x) - \left((-x)^3-5\right)(-1)$

7. $G(x)=\int_{x^3}^{5} \left(3t^2-7t+2\right)dt$

8. $G(x)=\int_1^{\sin x} 3t\,dt$

9. $G(x)=\int_0^x \frac{t}{\cos t}\,dt$

10. $G(x)=\int_0^{x^3+7x} 7\sin t\cos t\,dt$

→

Fundamental Theorem of Calculus, Part II:

IF f IS CONTINUOUS ON $[a, b]$ AND F IS AN ANTIDERIVATIVE OF f $\Longrightarrow$

$$\boxed{\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)}$$

Ex: $\int_1^4 x^2\,dx = \frac{1}{3}x^3\Big]_1^4 = \frac{1}{3}\left[4^3 - 1^3\right] = \frac{1}{3}(64-1) = \frac{1}{3}(63) = 21$

Review – Critical Points, Increasing and Decreasing

CRITICAL POINT: IF C IS IN DOMAIN OF $f(x)$ AND $f'(c) = 0$

OR $f'(c)$ IS UNDEFINED $\Rightarrow$ C IS A CRIT. PT OF $f(x)$.

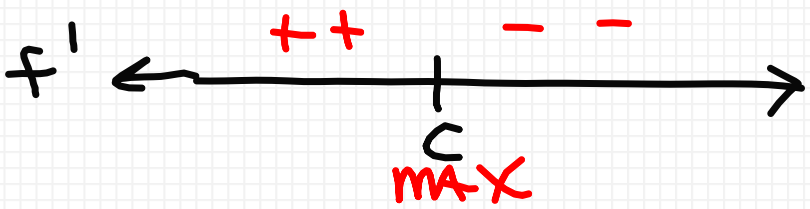
INCREASING: f IS INCR ON $[a,b]$ IF $f'(x) > 0$ ON (a,b)

DECREASING: f IS DECR ON $[a,b]$ IF $f'(x) < 0$ ON (a,b)

Review – First Derivative Test (Relative/Local Max and Min)

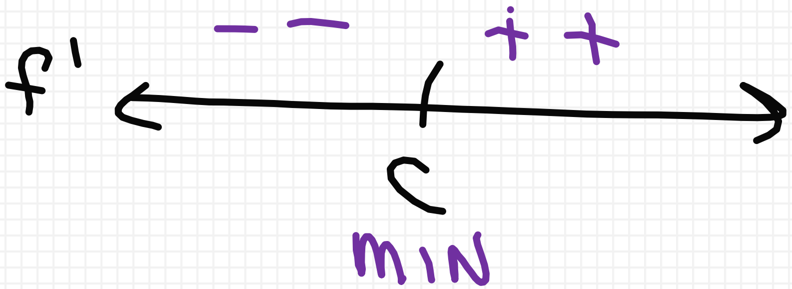
① IF C IS A CRIT. PT OF $f(x)$ AND f' CHANGES FROM POSITIVE TO NEGATIVE AT $x=c$

$\Rightarrow$ C IS A RELATIVE MAX OF $f(x)$.



② IF C IS A CRIT. PT OF $f(x)$ AND f' CHANGES FROM NEGATIVE TO POSITIVE AT $x=c$

$\Rightarrow$ C IS A RELATIVE MIN OF $f(x)$.



Review – Second Derivative Test (Relative/Local Max and Min)

① IF C IS CRIT PT OF $f(x)$ AND $f''(c) > 0$ $\Rightarrow$ C IS A RELATIVE MIN OF $f(x)$.

② IF C IS A CRIT. PT OF $f(x)$ AND $f''(c) < 0$ $\Rightarrow$ C IS A RELATIVE MAX OF $f(x)$.

Review – Concavity and Points of Inflection

① IF $f''(x) > 0$ on $(a,b) \Rightarrow f(x)$ IS CONCAVE UP on (a,b)

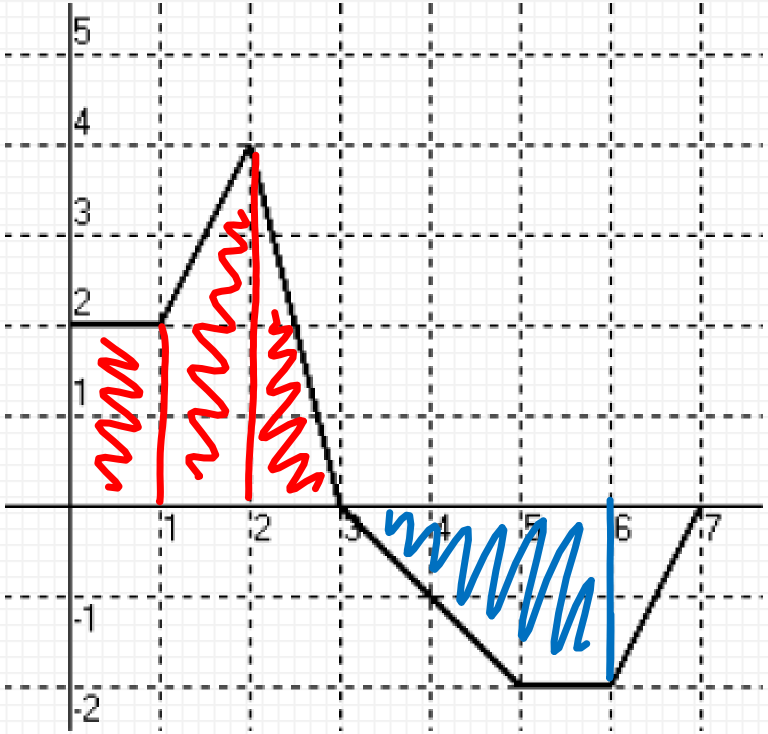
② IF $f''(x) < 0$ on $(a,b) \Rightarrow f(x)$ IS CONCAVE DOWN on (a,b)

③ POINT OF INFLECTION: MUST SHOW THAT $f''(x)$ <u>CHANGES</u>

FROM + TO − OR − TO + AT $x = c$.

AP Calculus BC
Section 5.3 – FTC Free Response Questions

1. (Stewart – no calculator) Let $g(x) = \int_0^x f(t)dt$, where f is the function whose graph is shown to the right.



a. Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(6)$.

$g(0) = \int_0^0 f(t)\,dt = 0$

$g(1) = \int_0^1 f(t)\,dt = 2$

$g(2) = \int_0^2 f(t)\,dt = 5$

$g(3) = \int_0^3 f(t)\,dt = 7$

$g(6) = \int_0^6 f(t)\,dt = 7-4$
$= 3$

b. On what intervals is g increasing?

$g' = f(x) > 0$ on $(0,3)$ $\Rightarrow$ g INCREASING on $(0,3)$.

c. Where does g have a maximum value? on $[0,7]$?

$g' = f = 0$ AT $x = 3$.

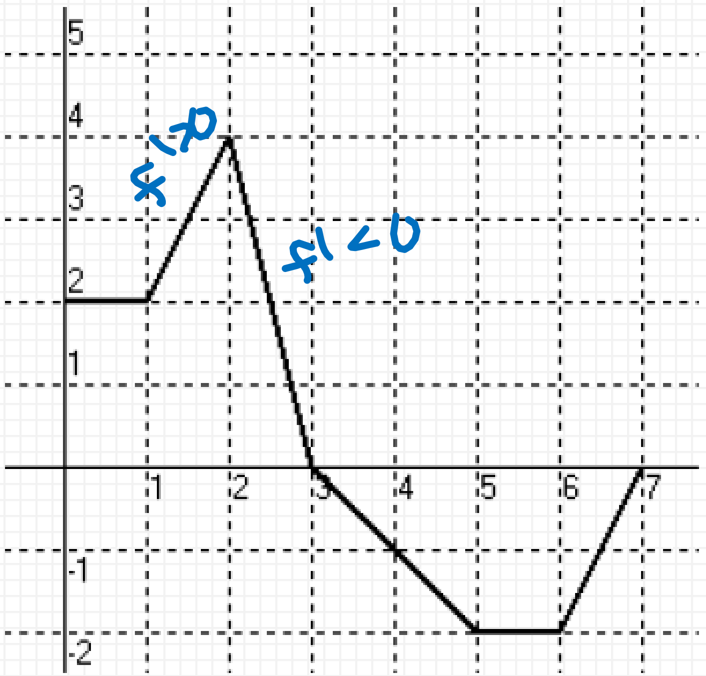
CANDIDATES TEST: $g(0) = 0$

$g(3) = 7$

$g(7) = 2$

g HAS A MAX VALUE AT $x=3$.

THE MAX VALUE IS 7.



d. Evaluate $g'(2)$

$g'(2) = f(2) = 4$.

e. Find any points of inflection. Justify your answers.

$g'' = f'$ CHANGES SIGN AT $x = 2$ $\Rightarrow$ POINT OF INFLECTION AT $x = 2$.

Homework:

Chapter 5 AP Packet #19-28
FTC FRQ #2,3